International Journal of Science, Engineering & Environmental Technology (IJOSEET), 10(11): 42-49, 2023
ISSN 0794-9650

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The Influence of Temperature on the Fluid Flow Pass a Loamy Soil with Magnetic Field, Using Spatial Dependent Thermal Conductivity

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Abstract – The influence of temperature on the fluid flow pass a loamy soil with spatial dependent thermal conductivity was investigated. The governing equations which were in dimensional form were reduced to non-dimensional form with the use of some dimensionless parameters. The resulted second order partial differential equations were further reduced to ordinary differential equations via perturbation method. The energy equation which is also found in the momentum equation was then solved alongside the momentum equation analytically. Basically, the resulting physical parameters, namely: the radiation parameter, the internal heat generation, Prandtl number and the thermal Grashof number were examined on the velocity of the fluid flow pass a loamy soil. When all the parameters are kept constant, as time goes on, the velocity of the fluid through the soil was also investigated. Numerical results were computed using Matlab R2009b and displayed on graphs. Increase in the intensity of the solar radiation, with a rising internal heat generation and thermal buoyancy force were discovered to speed up the rate of flow of fluid through the loamy soil, whereas the Prandtl number decreased the velocity at the boundary layer.

Keywords: Internal heat generation, loamy soil, perturbation method, Prandtl number, solar radiation, thermal Grashof number.

1. Introduction

Loamy soil is a commonly studied material due to its impact on numerous areas in the environment, ranging from agriculture and engineering, to water resources. They are soils that are composed of a balanced mixture of clay, sand and silt, making them highly fertile for plant growth. They have higher amount of organic matter than other types of soils, allowing plants to easily uptake nutrients and grow vigorously. The water-holding capacity of loamy soils is also higher than that of sand, which increases its fertility and allows it to retain moisture for longer periods of time. Moreover, its average porosity enables air to penetrate and create an environment of optimum plant growth. Due to its unique composition of clay, silt and sand, loamy soil tends to be more workable than other soil types, allowing for easier tilling and cultivation. Moreover, Loamy soil is more resilient against compaction and erosion, allowing for greater sustainability and reduces risk of nutrient loss. They are highly recommended for use in vegetable gardens and flower beds due to their ideal composition for horticultural use. In addition, loamy soil can be used for golf course fairways, sports turfs, and park lawns. However, there are various factors that can influence the rate of flow of fluids through the soil, and this obviously has ability to influence the usage of the soil for various purposes.

In their own work, Akinpelu *et al.* [1] combined clay with loam and sand with loam. They compared what the impact solar radiation would have on the temperature of both combinations. They thus reported a significant boost in the temperature of both combinations of the soils. Though, this boost was more in combination of sand and loam than in clay and loam. Not only that, the rate at which this boost occurred in the combination of sand and loam was also of greater measure than that of the clay and loam.

Likewise, putting into consideration, the Chemical reaction alongside heat generation, Mohammed [2] considered how radiation affects a mass flow via porous medium with high porosity. He found out that the flow rate was largely dependent on the parameter being in consideration as some of them like the internal heat, thermal and Solutal Grashof numbers speed up the flow rate. While others like radiation parameter, Prandtl number and Chemical parameter dragged the flow.

Focusing on the temperature of loamy soil, Olaleye *et al.* [3] examined how solar radiations shape the soil. They also considered the thermal conductivity to be spatial dependent and generated a Mathematical model for the problem. Examining some of the parameters that came up in the work, they established the fact that the solar radiation increases the soil temperature.

Furthermore, Alabison *et al.* [4] deliberated on how heat generation alongside radiation affect fluid flow passes a medium (porous) in a convectional way. As their boundary condition, they considered the temperature to be periodic. Their results evidently revealed that the buoyancy forces speed up the velocity of the fluid but the chemical reaction reduced the flow rate.

Among many others who have studied along this line of study are Mahender & Srikanth [5] who considered a flow through a straight up plate that is permeable in a convectional way in the presence of both the magnetic field as well as viscous dissipation, Sharma & Bismeeta [6] who delve into finding out the impact which some imperative physical parameters like Dufour and Soret numbers have on fluid flow through a straight up cone in free convective manner when the thermal conductivity and viscosity are varying, and Nwaigwe [7] who generally modeled ground temperature making the suction velocity constant.

This present work nonetheless significantly examines the influence that temperature has on the flow of fluid with magnetic field deploying thermal conductivity that is spatial dependent..

2. Mathematical Formulation

The model is formed under the assumptions that the flow is unsteady and the vertical axis (z – axis) being the only axis of concern. In other word, the flow is infinite down the horizontal axis (y-axis). Moreover, thesoil in consideration (loam) being porous is noted to also be optically thin; given a passage for heat to flow as well. To be more practically, at the surface of the soil, the temperature and suction velocity are taken to be variable. Following the usage of Boussinesq's approximation in existing literatures, the following leading equations are formulated for the work.

$$\frac{\partial w^*}{\partial z^*} = 0 \quad (1)$$

$$\frac{\partial v^*}{\partial t^*} + w^* \frac{\partial v^*}{\partial z^*} = \vartheta \frac{\partial^2 v^*}{\partial z^{*2}} + g\beta(T^* - T^*_\infty) - \frac{w}{K}\varphi v^* - \frac{\sigma B_0^2 v^*}{\rho} \quad (2)$$

$$\rho C_p \frac{\partial T^*}{\partial t^*} + \rho C_p w^* \frac{\partial T^*}{\partial z^*} = \frac{\partial}{\partial z}\left\{k \frac{\partial T^*}{\partial z^*}\right\} - \frac{\partial q_r^*}{\partial z^*} + Q_0(T^* - T^*_\infty) \quad (3)$$

Subject to:

$$v^* = V_p^*, \quad T^* = T_w^* + c^*(T_w^* - T_\infty^*)e^{i\omega^* t^*} \quad at \quad z^* = 0 \quad (4)$$

$$v^* = V_\infty^*, \quad T^* \to T_\infty^* \quad as \quad z^* \to \infty \quad (5)$$

All the equations are in dimensional form; which are the continuity (equation 1), momentum (equation 2) and energy equations (equation 3) respectively. Moreover, equations (4) & (5) are the boundary conditions.

3. Further Analysis

Following some published works, like Akinpelu *et al.* [1], suction velocity (varying with time) is known to be:

$$w^* = -w_0\left(1 + \varepsilon A e^{i\omega^* t^*}\right) \quad (6)$$

where

w_0, A and ω^* respectively are the initial suction velocity, suction parameter and frequency of oscillation.

Furthermore, in agreement with the some existing literatures like Nwaigwe [7], the thermal conductivity is taken to be spatial dependent, and given as:

$$k = k_0(1 + bz) \quad (7)$$

where,

k_0 = the constant thermal conductivity

b = parameter of the variable thermal conductivity

The radiative heat flux is also defined to be:

$$\frac{\partial q_r^*}{\partial z^*} = 4\alpha^2(T^* - T_\infty^*) \quad (8)$$

α = absorption coefficient.

In agreement with Mohammed [2] and some other literatures, the following non-dimensional quantities were used:

$$\omega = \frac{w\omega^*}{w_0^2}, \ z = \frac{w_0 z^*}{w}, \ v = \frac{v^*}{V_0}, \ t = \frac{t^* w_0^2}{w}, \ c = \frac{c^*}{T_W^* - T_\infty^*}$$

$$Gr = \frac{vg\beta(T_W^* - T_\infty^*)}{w_0^3}, \ P_r = \frac{v\rho C_p}{k_\infty}, \ \theta = \frac{T^* - T_\infty^*}{T_W^* - T_\infty^*}, \ M = \frac{\sigma B_0^2 w}{\rho w_0^2}, \ K = \frac{K^* w_0^2}{\varphi w^2} \quad (9)$$

Using equations (6) – (9), equations (2) – (5) were reduced to:

$$v'(t) - v'(z) - \varepsilon A e^{i\omega t} v'(z) = v''(z) + Gr\theta - \left[M + \frac{1}{K}\right] v \quad (10)$$

$$\theta'(t) - \theta'(z) - \varepsilon A e^{i\omega t} \theta'(z) = \frac{1}{P_r}\theta''(z) + bz\theta''(z) - R^2 + Q\theta \quad (11)$$

Subject to:

$$v = V_p, \ \theta = 1 + ce^{i\omega t} \quad at \quad z = 0 \quad (12)$$

$$v \to 1, \ \theta \to 0 \quad as \quad z \to \infty \quad (13)$$

where P_r = Prandtl number

Gr = thermal Grashof number

b = thermal conductivity parameter

R = radiation parameter and

Q = heat generation parameter.

4. Method of Solution

Reducing equations (10) – (11) to ordinary differential equations with the use of perturbation method, we assumed the solutions to be:

$$v(z,t) = v_0(z) + \varepsilon e^{i\omega t} v_1(z) + \varepsilon^2 e^{2i\omega t} v_2(z) + \cdots \quad (14)$$

$$\theta(z,t) = \theta_0(z) + \varepsilon e^{i\omega t} \theta_1(z) + \varepsilon^2 e^{2i\omega t} \theta_2(z) + \cdots \quad (15)$$

Ignoring higher order terms $o(\varepsilon^2)$, equations (10) – (11) by equations (14) – (15) became:

$$v_0''(z) + v_0'(z) - \frac{1}{K} v_0 = -Gr\theta_0 \quad (16)$$

$$v_1''(z) + v_1'(z) - \left(i\omega + \frac{1}{K}\right) v_1 = -Av_0'(z) - Gr\theta_1 \quad (17)$$

$$\theta_0''(z) + bz\theta_0''(z) + P_r\theta_0'(z) + P_r Q\theta_0 = P_r R^2 \quad (18)$$

$$\theta_1''(z) + bz\theta_1'(z) + P_r\theta_1'(z) + P_r(Q - i\omega)\theta_1 = -P_r A\theta_0'(z) \quad (19)$$

Subject to:

$$v_0 = V_p, \ v_1 = 0, \ \theta_0 = 1, \ \theta_1 = 1 \quad at \quad z = 0 \quad (20)$$

$$v_0 \to 1, \ v_1 \to 0, \ \theta_0 \to 0, \ \theta_1 \to 0 \quad as \quad z \to \infty \quad (21)$$

Following Umavathi [8], second phase perturbation method for equations (18) – (19) is given as:

$$\theta_0 = \theta_{00} + b\theta_{01} \quad (22)$$
$$\theta_1 = \theta_{10} + b\theta_{11} \quad (23)$$

Ignoring higher order $o(b^2)$ also, equations (18) – (19) by equations (22) – (23) became:

$$\theta_{00}''(z) + P_r\theta_{00}'(z) + P_rQ\theta_{00} = P_rR^2 \quad (24)$$
$$\theta_{01}''(z) + P_r\theta_{01}'(z) + P_rQ\theta_{01} = -z\theta_{00}''(z) \quad (25)$$
$$\theta_{10}''(z) + P_r\theta_{10}'(z) + P_r(Q - i\omega)\theta_{10} = -P_rA\theta_{00}'(z) \quad (26)$$
$$\theta_{11}''(z) + P_r\theta_{11}'(z) + P_r(Q - i\omega)\theta_{11} = -z\theta_{10}''(z) - P_rA\theta_{01}'(z) \quad (27)$$

The solutions of equations (24) – (27) then became,

$$\theta_{00} = C_1e^{m_1z} + C_2e^{m_2z} + C_3 \quad (28)$$
$$\theta_{01} = C_5e^{m_4z} + (C_6z + C_7)e^{m_2z} + (C_8z + C_9)e^{(m_2-m_1)z} \quad (29)$$
$$\theta_{10} = C_{11}e^{m_6z} + C_{12}e^{m_2z} + C_{13}e^{(m_2-m_1)z} \quad (30)$$
$$\theta_{11} = C_{13}e^{m_8z} + (C_{14}z + C_{15})e^{m_2z} + C_{16}e^{m_4z} + (C_{17}z + C_{18})e^{m_6z} + (C_{19}z + C_{20})e^{(m_2-m_1)z} \quad (31)$$

And the solutions of equations (16) – (17) also became,

$$v_0 = C_{13}e^{m_9z} + C_{14}e^{m_{10}z} + C_{15}e^{m_1z} + C_{16}e^{m_2z} + C_{17}e^{m_3z} + C_{18}e^{m_4z} + C_{19} \quad (32)$$
$$v_1 = C_{20}e^{m_{11}z} + C_{21}e^{m_{12}z} + C_{22}e^{m_9z} + C_{23}e^{m_{10}z} + C_{24}e^{m_1z} + C_{25}e^{m_2z} + C_{26}e^{m_3z} + C_{27}e^{m_4z} + C_{28}e^{m_5z} + C_{29}e^{m_6z} + C_{30}e^{m_7z} + C_{31}e^{m_8z} + C_{32} \quad (33)$$

So, we write the velocity and temperature distributions as:

$$v = v_0 + \varepsilon e^{i\omega t}v_1 \quad (34)$$
$$\theta = \theta_{00} + \varepsilon e^{i\omega t}\theta_{10} + b(\theta_{01} + \varepsilon e^{i\omega t}\theta_{11}) \quad (35)$$

5. Results and Discussion

In order to achieve the aim of the study, the emerged physical parameters were examined on the velocity of the fluid flow which represented as equation (34). The thermal conductivity and the permeability of loamy soil given to be 0.52 Btu/ft hr ^{0}F [9] and 1.30cm/hour [10] respectively were used.

Moreover, some other default parameters were also used which were mainly adopted from existing literatures and given below.

Table 1. Other parameters used with their default values

R	Gr	t	Q	V_p	A	ω	c	P_r	ε
0.50	5.00	1.00	0.10	10.00	0.50	$\pi/2$	1.00	0.71	0.01

All the values remain the same except it is stated otherwise.

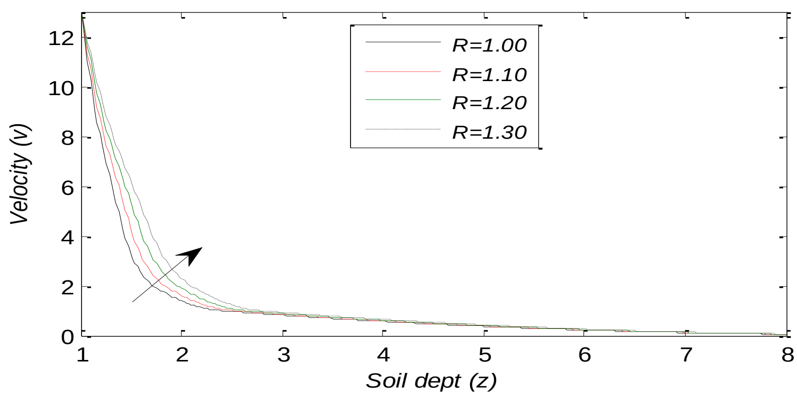

Fig. 1. The result of rising R on the rate of fluid flow passes loamy soil

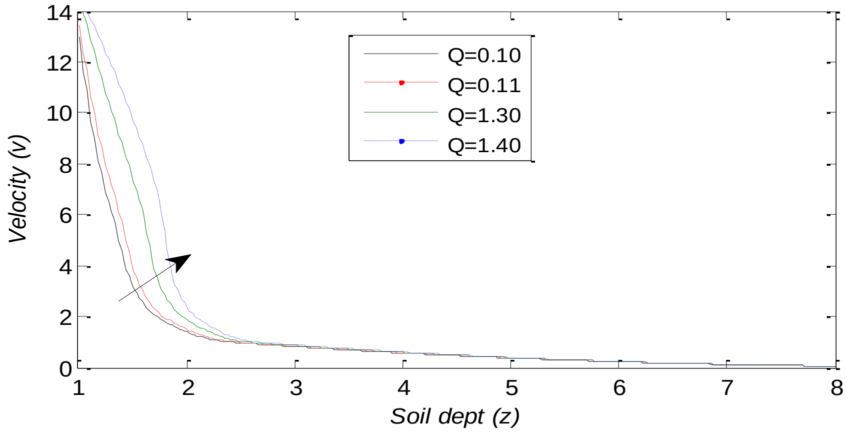

Fig. 2. The result of rising Q on the rate of fluid flow passes loamy soil

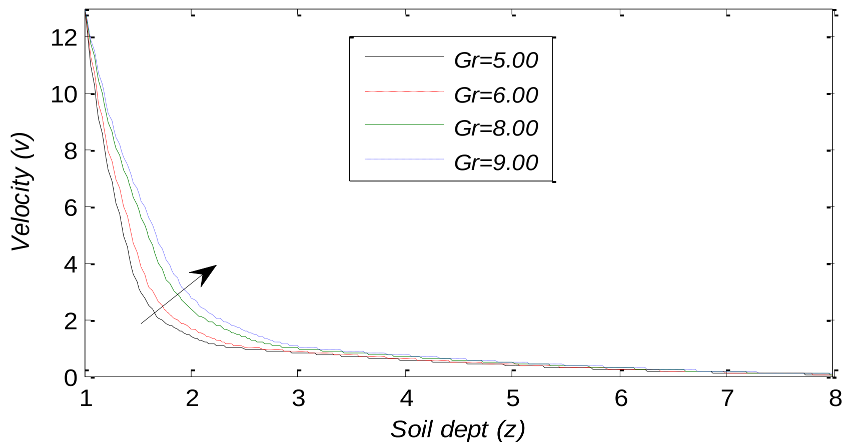

Fig. 3. The result of rising Gr on the rate of fluid flow passes loamy soil

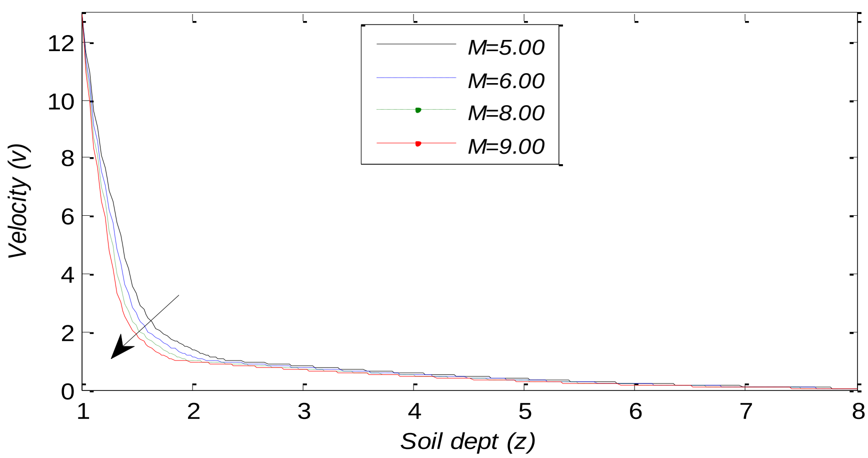

Fig. 4. The result of rising M on the rate of fluid flow passes loamy soil

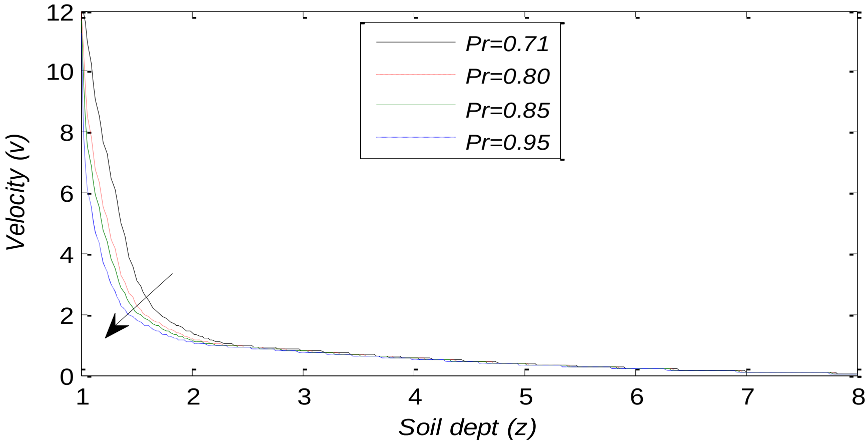

Fig. 5. The result of rising Pr on the rate of fluid flow passes loamy soil

Figure 1 is the influence of escalating radiation on the fluid's velocity via loamy soil. It is clear that the flow rate is increasing as the intensity of the radiation also increases. Also, according to figure 2, when the internal heat increases, the fluid flows at a faster rate. Moreover, in Figure 3 which depicts the influence of thermal Grashof number on the flow velocity of the fluids through the sample soil. It is evident that an increasing thermal buoyancy force enhances the velocity of the fluid.

In figure 4, the Lorentz force is seen in action as it drags the rate of flow of the fluid because of the increasing magnetic field parameter. Figure 5 shows the effect of Prandtl number (Pr) on the velocity of fluid's flow through the loamy soil. An increasing Prandtl number obviously decreases the velocity of the flow.

6. Conclusion

Influence of temperature on the fluid flow pass a loamy soil with magnetic field, using spatial dependent thermal conductivity has been investigated. The model was formed using appropriate equations and assumptions. The effects of the physical parameters that came up were examined on the velocity of the fluid flow. It was discovered that the increase in temperature through various means which include radiation, internal heat and thermal buoyancy force, increase the speed of the fluid flow through loamy soil. However, when the magnetic field increases, the rate of flow is reduced.

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Nomenclatures

$z^* = vertical\ axis$, $t^* = time$, $u^* = component\ velocity\ along\ z^*$

$w^* = suction\ velocity$, $T^* = Temperature$, $T_w^* = Wall\ temperature$

$T_\infty^* = Free\ stream\ temperature$, $\vartheta = Kinematic\ viscosity$, $g = acceleration\ due\ to\ gravity$

$V_P^* = Wall\ velocity$, $V_\infty^* = Free\ sream\ velocity$, $\rho = Density$

$\beta = Volumetric\ coefficient\ of\ thermal\ expansion$, $C_p = Specific\ heat\ capacity$

$K^* = Soil\ permeability$, $k = Thermal\ conductivity$, $\varphi = Porosity$

$c = Amplitude\ of\ variation$, $q_r^* = Radiative\ heat\ flux$

Appendix

$$m_1 = -\frac{P_r}{2} + \sqrt{\frac{P_r^2}{4} - P_r Q} \qquad m_2 = -\left(\frac{P_r}{2} + \sqrt{\frac{P_r^2}{4} - P_r Q}\right) \qquad m_3 = -\frac{P_r}{2} + \sqrt{\frac{P_r^2}{4} - P_r Q}$$

$$m_4 = -\left(\frac{P_r}{2} + \sqrt{\frac{P_r^2}{4} - P_r Q}\right) \qquad m_5 = -\frac{P_r}{2} + \sqrt{\frac{P_r^2}{4} - P_r(Q - i\omega)} \qquad m_6 = -\left(\frac{P_r}{2} + \sqrt{\frac{P_r^2}{4} - P_r(Q - i\omega)}\right)$$

$$m_7 = -\frac{P_r}{2} + \sqrt{\frac{P_r^2}{4} - P_r(Q - i\omega)} \qquad m_8 = -\left(\frac{P_r}{2} + \sqrt{\frac{P_r^2}{4} - P_r(Q - i\omega)}\right) \qquad m_9 = -\frac{1}{2} + \sqrt{\frac{K+4MK+4}{4K}}$$

$$m_{10} = -\left(\frac{1}{2} + \sqrt{\frac{K+4MK+4}{4K}}\right) \qquad m_{11} = -\frac{1}{2} + \sqrt{\frac{K+4MK+4+4i\omega K}{4K}} \qquad m_{12} = -\left(\frac{1}{2} + \sqrt{\frac{K+4MK+4+4i\omega K}{4K}}\right)$$

$$C_1 = -C_3 e^{-m_1 z} \qquad C_2 = 1 + \tau e^{i\omega t} - C_1 - C_3 \qquad C_3 = \frac{R^2}{Q} \qquad C_5 = -(C_7 + C_9)$$

$$C_6 = \frac{(C_3 - 1 - \tau e^{i\omega t}) m_2^2}{m_2^2 + P_r m_2 + P_r Q} \qquad C_7 = \frac{-2m_2 C_6 - P_r C_6}{m_2^2 + P_r m_2 + P_r Q} \qquad C_8 = \frac{(m_1 m_2 + m_1(m_2 - m_1) - m_2^2) C_3}{(m_2 - m_1)^2 + P_r(m_2 - m_1) + P_r Q}$$

$$C_9 = \frac{-2(m_2 - m_1)C_3 - P_r C_3}{(m_2 - m_1)^2 + P_r(m_2 - m_1) + P_r Q}, \qquad C_{11} = -(C_{12} + C_{13}) \qquad C_{12} = \frac{P_r A m_2 (C_3 - 1 - \tau e^{i\omega t})}{m_2^2 + P_r m_2 + P_r(Q - i\omega)}$$

$$C_{13} = \frac{-P_r A(m_2 - m_1)C_3}{(m_2 - m_1)^2 + P_r(m_2 - m_1) + P_r(Q - i\omega)} \qquad C_{14} = \frac{-m_2^2 C_{12} - P_r A m_2 C_6}{m_2^2 + P_r m_2 + P_r(Q - i\omega)} \qquad C_{15} = \frac{-P_r A(m_2 C_7 + C_6) - 2m_2 C_{14} - P_r C_{14})}{m_2^2 + P_r m_2 + P_r(Q - i\omega)}$$

$$C_{16} = \frac{-P_r A m_4 C_5}{m_4^2 + P_r m_4 + P_r(Q - i\omega)} \qquad C_{17} = \frac{-m_6^2 C_{11}}{m_6^2 + P_r m_6 + P_r(Q - i\omega)} \qquad C_{18} = \frac{-2m_6 C_{17} - P_r C_{17}}{m_6^2 + P_r m_6 + P_r(Q - i\omega)}$$

$$C_{19} = \frac{-(m_2 - m_1)^2 C_{13} - P_r A(m_2 - m_1)C_8}{(m_2 - m_1)^2 + P_r(m_2 - m_1) + P_r(Q - i\omega)} \qquad C_{20} = \frac{-P_r A((m_2 - m_1)C_9 + C_8) - 2(m_2 - m_1)C_{19} - P_r C_{19}}{(m_2 - m_1)^2 + P_r(m_2 - m_1) + P_r(Q - i\omega)}$$

$$C_{21} = (1 - C_{27} - C_{23} e^{m_1 z} - C_{25} e^{m_3 z}) e^{-m_9 z} \qquad C_{22} = V_p - C_{21} - C_{23} - C_{24} - C_{25} - C_{26} - C_{27}$$

$$C_{23} = \frac{-GrC_1}{m_1^2 + m_1 - \left(\frac{MK+1}{K}\right)} \qquad C_{24} = \frac{-GrC_2}{m_2^2 + m_2 - \left(\frac{MK+1}{K}\right)} \qquad C_{25} = \frac{-GrDC_4}{m_3^2 + m_3 - \left(\frac{MK+1}{K}\right)}$$

$$C_{26} = \frac{-GrDC_5}{m_4^2 + m_4 - \left(\frac{MK+1}{K}\right)} \qquad C_{27} = \frac{GrK(C_3 + DC_6)}{MK+1}$$

$$C_{28} = -(C_{30} e^{m_9 z} + C_{32} e^{m_1 z} + C_{34} e^{m_3 z} + C_{36} e^{m_5 z} + C_{38} e^{m_7 z} + C_{40}) e^{-m_{11} z}$$

$$C_{29} = -(C_{28} + C_{30} + C_{31} + C_{32} + C_{33} + C_{34} + C_{35} + C_{36} + C_{37} + C_{38} + C_{39} + C_{40})$$

$$C_{30} = \frac{-Am_9 C_{21}}{m_9^2 + m_9 - \left(i\omega + \frac{MK+1}{K}\right)} \qquad C_{31} = \frac{-Am_{10} C_{22}}{m_{10}^2 + m_{10} - \left(i\omega + \frac{MK+1}{K}\right)} \qquad C_{32} = \frac{-Am_1 C_{23}}{m_1^2 + m_1 - \left(i\omega + \frac{MK+1}{K}\right)}$$

$$C_{33} = \frac{-Am_2 C_{24}}{m_2^2 + m_2 - \left(i\omega + \frac{MK+1}{K}\right)} \qquad C_{34} = \frac{-Am_3 C_{25}}{m_3^2 + m_3 - \left(i\omega + \frac{MK+1}{K}\right)} \qquad C_{35} = \frac{-Am_4 C_{26}}{m_4^2 + m_4 - \left(i\omega + \frac{MK+1}{K}\right)}$$

$$C_{36} = \frac{-GrC_7}{m_5^2 + m_5 - \left(i\omega + \frac{MK+1}{K}\right)} \qquad C_{37} = \frac{-GrC_8}{m_6^2 + m_6 - \left(i\omega + \frac{MK+1}{K}\right)} \qquad C_{38} = \frac{-GrDC_{10}}{m_7^2 + m_7 - \left(i\omega + \frac{MK+1}{K}\right)}$$

$$C_{39} = \frac{-GrDC_{11}}{m_8^2 + m_8 - \left(i\omega + \frac{MK+1}{K}\right)} \qquad C_{40} = \frac{GrK(C_9 + DC}{i\omega K + MK +}$$